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On an Hermite–Birkhoff Problem of Passow

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In a recent paper [6] Passow considers incidence matrices $E = (e_{ij})_{i=1, j=0}^{N \ \ n}$, which in their first column contain exactly two ones. He then relates the poisedness of E on a particular set of points $X = \{x_1, ..., x_N\}$ to the existence of certain quadrature formulas and in particular notes that if there exists a quadrature formula of precision $n-1$ based on X then the problem $(E, X, \mathscr{P}_n)$ is not poised. In this work we confirm Passow's conjecture that the converse is also true, extending its scope.

We take the following viewpoint of this result. Recall that E is certainly not poised unless it satisfies the Pòlya conditions $M_l = \sum_{j=0}^{l} \sum_{i=1}^{N} e_{ij} \geqslant l+1$, $l = 0, ..., n-1$, $M_n = n+1$; see [7]. If $M_l = l+1$ for some l then E is decomposable and poised if and only if each element of the decomposition is poised [1]. Thus E may be assumed to satisfy the Birkhoff condition $M_l \geqslant l+2$, $l = 0, ..., n-1$. If now there is equality in the case $M_{k-1} = k+1$ then one may ask whether it is possible to determine the poisedness of E on the basis of the properties of its components

$$\begin{aligned} F &= (f_{ij})_{i=1, j=0}^{N \ \ k}, \qquad f_{ij} = e_{ij}, \quad j < k; \quad f_{ik} = 0, \\ L &= (l_{ij})_{i=1, j=0}^{N \ \ n-k-1}, \qquad l_{ij} = e_{i, j+k}. \end{aligned} \tag{1}$$

For this purpose we need some results dating back to Birkhoff [2]; see also [4]. Denote

$$D_F(X) = \det \left| \frac{x_i^{n-k}}{(n-k)!}, ..., \frac{x_i^{-k}}{(-k)!} \right| \text{ over } f_{ik} = 1, \text{ where } (r!)^{-1} = 0 \text{ for } r < 0,$$

so that $(F, X, \mathscr{P}_k)$ is poised if and only if $D_F(X) \neq 0$. Let $D_{ij}(X)$ be the algebraic complements of the first column of $D_F(X)$. Thus $D_{ij}(X) = D_{F_{ij}}(X)$,

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where F_{ij} is obtained from F by changing entry f_{ij} from 1 to 0. Birkhoff's kernel is given by

$$K_F(X, t) = \det \left| \frac{(x_i - t)_+^{k-j-1}}{(k-j-1)!}, \frac{x_i^{k-j-1}}{(k-j-1)!}, \ldots, \frac{x_i^{-j}}{(-j)!} \right|, \quad f_{ij} = 1. \tag{2}$$

We have for any $g \in C^k[x_1, x_N]$

$$\sum_{f_{ij}=1} D_{ij}(X)\, g^{(j)}(x_i) = \int_{x_1}^{x_n} K_F(X, t)\, g^{(k)}(t)\, dt. \tag{3}$$

Furthermore if F contains no odd supported blocks [5] then it is well known [1] that $D_F(X) \neq 0$. As a result for such F

(a) $K_F(X, t) \geqslant 0$ everywhere,

(b) $D_{ik}(X) \neq 0, f_{ik} = 1$, for F_{ik} again has no odd supported blocks.

THEOREM 1. *Suppose F contains no odd supported blocks. Then $(E, X, \mathscr{P}_n)$ is not poised if and only if there exists $q \in \mathscr{P}_{n-k}$ annihilating (L, X) and such that*

$$\int_{x_1}^{x_N} K_F(X, t)\, q(t)\, dt = 0. \tag{4}$$

Proof. Suppose $(E, X, \mathscr{P}_n)$ is not poised, let $p \in \mathscr{P}_n$ annihilate (E, X) and denote $q = p^{(k)}$. Then q annihilates (L, X) and by identity (3) $\int_{x_1}^{x_N} K_F(X, t)\, q(t)\, dt = 0$.

For the converse, let $H(y, t) = K_{\tilde{F}}(\tilde{X}, t)$, where $\tilde{X} = X \cup \{y\}$ and $\tilde{F}$ is obtained from F by changing the non-zero entry $f_{r,0}$ from 1 to 0 and adding a row corresponding to y, with a single non-zero entry in the 0 column. Now given q define

$$p(y) = \int_{x_1}^{x_N} H(y, t)\, q(t)\, dt.$$

A look at (2) shows $p^{(k)}(y) = D_{r,0}(X)\, q(y)$, $p^{(j)}(x_i) = 0, f_{ij} = 1$, $(i, j) \neq (r, 0)$ and thus identity (3) becomes

$$D_{r,0}\, p(x_r) = \int_{x_1}^{x_N} K_F(X, t)\, q(t)\, dt,$$

which vanishes by assumption. Hence p annihilates (E, X).

The condition of Theorem 1 is sometimes easy to test.

EXAMPLE. In the spirit of Lorentz and Zeller [5] consider

$$E = \begin{matrix} -1 \\ x_2 \\ 1 \end{matrix} \left| \begin{matrix} 1 & 1 & . & . & \overset{k}{1} & 0 & . & . & 0 \\ 0 & 1 & 0 & . & . & . & 0 & 1 & 0 \\ 1 & 1 & . & . & 1 & 0 & . & . & 0 \end{matrix} \right|, \qquad L = \begin{matrix} -1 \\ x_2 \\ 1 \end{matrix} \left| \begin{matrix} 1 & 1 & . & . & \overset{k-1}{1} & 0 & . & 0 \\ 1 & 0 & . & . & . & . & 0 & 1 \\ 1 & 1 & . & . & 1 & 0 & . & 0 \end{matrix} \right|.$$

Note that $(L, X, \mathscr{P}_{2k+1})$ is poised for all X and hence $p \in \mathscr{P}_{2k+2}$ annihilating (L, X) is unique up to a multiplicative constant, $p(t) = (t^2 - 1)^k (t - x_2) \cdot (t - (2k+1)\,x_2)$. Now the equation

$$I_{k+1} + (1 + (2k+1)\,x_2^2)\,I_k = \int_{-1}^{1} p(t)\,dt = 0$$

has no real solutions since

$$I_{k+1} = \int_{-1}^{1} (t^2 - 1)^{k+1}\,dt = -\frac{2k+3}{2k+3}\,I_k.$$

Thus $(E, X, \mathscr{P}_n)$ is always poised.

THEOREM 2. *Suppose F contains no odd supported blocks and $(L, X, \mathscr{P}_{n-k-1})$ is poised. Then $(E, X, \mathscr{P}_n)$ is not poised if and only if there exists a quadrature formula of the form*

$$\int_{x_1}^{x_N} K_F(X, t)\,g(t)\,dt \approx \sum_{l_{ij}=1} a_{ij}\,g^{(j)}(x_i), \tag{5}$$

which is exact for polynomials of degree $n - k$.

Proof. There always exists $q \in \mathscr{P}_{n-k}$ annihilating (L, X) since only $n - k$ conditions are specified. If (5) is exact for $\mathscr{P}_{n-k}$ then clearly (4) holds and hence $(E, X, \mathscr{P}_n)$ is not poised. For the converse note that since $(L, X, \mathscr{P}_{n-k-1})$ is poised it is possible to interpolate with $\mathscr{P}_{n-k-1}$ to the data $g^{(j)}(x_i)$, $l_{ij} = 1$. Hence (5) may be assumed to be interpolatory and exact at least for $\mathscr{P}_{n-k-1}$.

Let $p \in \mathscr{P}_n$ annihilate (E, X). Then $q = p^{(k)}$ satisfies (4) and since q annihilates (L, X) but $(L, X, \mathscr{P}_{n-k1})$ is poised, the degree of q must be exactly $n - k$. Therefore any $Q \in \mathscr{P}_{n-k}$ may be written as $Q(t) = Aq(t) + r(t)$, where $r \in \mathscr{P}_{n-k-1}$ and hence

$$\int_{x_1}^{x_N} K_F(X, t)\, Q(t)\, dt$$

$$= \int_{x_1}^{x_N} K_F(X, t)\, r(t)\, dt$$

$$= \sum_{l_{ij}=1} a_{ij} r^{(j)}(x_i) = \sum_{l_{ij}=1} a_{ij} [Aq^{(j)}(x_i) + r^{(j)}(x_i)].$$

Remark 1. If F contains no odd supported blocks then $K_F(X, t)$ is a non-negative weight function. Thus results analogous to Passow's may be derived. Specifically, if all supported blocks in L are even, and

$$\int_{x_1}^{x_N} K_F(X, t)\ g(t)\, dt \approx \sum_{i=1}^{M} b_i\, g(y_i)$$

is a quadrature formula exact for $\mathscr{P}_{n-k}$, $b_i > 0$ $i = 1,..., M$, and the number of Hermite conditions in (y_i, y_j) $j = 2,..., M$ is always even, then $(E, X, \mathscr{P}_n)$ is poised.

Remark 2. Many questions are left unanswered. For example, what happens if $(F, X, \mathscr{P}_k)$ is not poised? It is easy to see that $(E, X, \mathscr{P}_n)$ may still be poised. Or: is it possible to come up with a nice condition if $(L, X, \mathscr{P}_{n-k-1})$ is not poised? It is clear that if the co-rank of L is 2, see [3], then $(E, X, \mathscr{P}_n)$ is not poised.

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